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Space-Time Structure and Measurement Theory

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Abstract

The present paper is a naïve operational approach to measurement theory in a truly relativistic framework. Both experiments and states exist in finite regions of space-time. The causality structure of the underlying Minkowski space is described in terms of these.

1. *Introduction*

The last few years have witnessed a renewed interest in the quantum theory of measurement. This revival has centered around the justification or critique of the 'orthodox' theory of von Neumann (1955). Numerous papers, even books, have been devoted to the subject, from points of view ranging from the most down-to-earth to the most philosophical.§ We have no intention to join this raging controversy, rooted as it is in the deepest epistemological problems of physics.

However, it is remarkable that nearly all the papers in measurement theory are in essence non-relativistic. A measurement is usually defined as an instantaneous act, or, if time duration is allowed, the process is considered on an absolute time scale (e.g. the widespread use of repreated experiments). As far as we know, only two papers—one by Schlieder (1968) and one by Hellwig & Kraus (1970)—depart from that position. They try to build a genuine relativistic theory of measurement using, however, the conventional language of density matrices.

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§ See for instance, Daneri *et al.* (1962), d'Espagnat (1965), Jauch *et al.* (1967), Jasselette (1970) and Jasselette *et al.* (1970).

On the other hand, another approach has been recently proposed which is operational in character. The idea is to characterize actual experiments, to see what experiments really do, and then to abstract from this a consistent mathematical scheme. Typically these authors, such as Mielnik (1968, 1969), Davies & Lewis (1970) or Davies (1970), approach the problem totally open-minded, not prejudiced in favor of a given structure such as Hilbert space or lattice theory ('quantum logic').

The present paper is an attempt to give a naïve, operational approach to measurement theory in a truly relativistic (i.e. space-time) framework. There are two basic ideas. First, an actual experiment, i.e. not a 'gedanken' experiment but one realizable in a laboratory, always takes place in a finite region of space. The equipment is turned on for a finite duration of time. Thus, actual experiments take place in finite regions of space-time. This is consistent with the fact that all realistic forces (strong, weak, effective electromagnetic) except gravitation (which is the subject of general relativity) have a finite range.

Secondly, the objects which are subjected to a measurement, or more generally, to an interaction, are by their very definition free outside that region of interaction. This of course does not mean that there are no correlations among these free objects. If a particle decays into two particles, after these two leave the interaction region they move freely despite their obvious correlations due to conservation laws. In the language of scattering theory, we say that 'asymptopia' begins right outside the finite range of the interaction.

Further, the objects of our theory have, at any time during their free evolution, a finite extent in space. In a space-time diagram, they look like a 'beam', propagating at a speed not exceeding the speed of light. Here, obviously, we are in conflict with the usual assumptions of local quantum field theory (Haag & Kastler, 1964) where observables are usually localized but states are not. Here, we take both observables (experiments) and states to be localized in space.

Now, an experiment may be described as follows: free 'beams' of 'particles' enter an interaction region R and other 'beams' emerge from R. If there are pieces of equipment located in R, they might possibly register a finite number of numerical results. Thus, in order to describe such a process, what is needed is a definition of a 'beam', an 'interaction region' and an 'experiment'. This is done in Section 2.

In Section 3 we list a few axioms which are essentially of the common-sense type. Section 4 is devoted to the causality structure of the theory. We establish a one-to-one correspondence between the mutual geometric positions of two interaction regions and a relation between all experiments that can be performed in the two regions in question. This is achieved via the concept of transparency. A collection of experiments E is, roughly speaking, transparent to another collection E' if no experiment of E' can influence an experiment of E. We then establish a link between the transparency relation and the natural partial order of space-time.

Finally, Section 5 deals with the Poincaré invariance of the theory. The main result is an analogue of Zeeman's theorem (Zeeman, 1964). It says, essentially, that any one-to-one transformation of Minkowski space that preserves all transparency relations is a conformal Poincaré transformation (Poincaré transformation with a dilation). We thus end with a purely experimental statement which is independent of any particular dynamical theory (classical or quantum mechanical, linear or non-linear). Indeed, the formalism is completely general and must be viewed as a set of consistency relations between a physical theory of measurement on the one hand and the geometry of Minkowski space and the principle of Einstein causality on the other.

The number of symbols and definitions in this paper is quite large. To aid the reader, we have included a list of symbols after the main text. In these, we include certain standard symbols which might be unfamiliar to the reader. Also after the main text, we have listed all the axioms in one place for easy reference.

2. *General Definitions*

We shall work consistently in Minkowski space, which will be denoted by $\mathscr{M}$.

2.1. *Interaction Regions*

A three-dimensional submanifold S of $\mathscr{M}$ is called a *space-like surface* if the distance between any two points in S is space-like. We denote by $\mathscr{S}$ the collection of all space-like surfaces. A *straight world line* is a geodesic† in $\mathscr{M}$ whose tangent is at each point either time-like or light-like. Using these notions, we call a collection T of straight world lines a *world tube* (or simply a *tube*) if

(1) T is the closure of its interior.
(2) For each $S \in \mathscr{S}$, $S \cap T$ is convex and compact.

It follows easily that any tube is arc-wise connected. We shall denote by $\mathscr{T}$ the collection of all tubes.

We have said before that 'particles' are free outside interaction regions. Hence, their trajectories are geodesics. Note that the 'particles' may be at rest, such as a crystal or a target in a scattering experiment.

We call an element (T, S_1, S_2) of $\mathscr{T} \times \mathscr{S} \times \mathscr{S}$ *admissible* iff

$$T \cap S_1 < T \cap S_2$$

where the sign '$<$' refers to the time ordering in $\mathscr{M}$. We associate to each admissible set (T, S_1, S_2) an *interaction region* R by

$$R = \{x \in \mathscr{M} | x \in T,\ T \cap S_1 \leqslant x \leqslant T \cap S_2\}$$

† A curve in a manifold whose tangent vector field is parallel along itself; in $\mathscr{M}$, straight line and geodesic are the same.

Here, again, $\leqslant$ refers to the time ordering in $\mathscr{M}$. We shall write $\mathscr{R}$ for the collection of all interaction regions. We will write $R = (T, S_1, S_2)$ if R is the interaction region associated with the admissible set (T, S_1, S_2). R is just the section of the tube T bounded in time by the two surfaces S_1 and S_2. We let

$$\mathrm{Beg}(R) = T \cap S_1$$

and

$$\mathrm{End}(R) = T \cap S_2$$

which denote the *beginning* and the *end* of R, respectively. We note that each interaction region R has interior, is compact, and is the closure of its interior.

2.2. *States*

We now come to the description of the states. First, let us call $\mathscr{C}$ some undetermined set of quantum numbers (usually discrete) which completely describes the composition of the free physical states. $\mathscr{C}$ might include masses, spins, and charges, among others.

Let B_1 be a subset of all quadruples of the form

$$b = (T, C, \lambda, S)$$

with $T \in \mathscr{T}$, $C \in \mathscr{C}$, $\lambda \in [0, \infty)$, and $S \in \mathscr{S}$, where we identify all quadruples with $\lambda = 0$. The corresponding element is called the *vacuum* and is denoted by Ω. It is obviously unique. An element $b \in B_1$ can be called a *single beam*; it propagates in the tube T with composition C and intensity λ. We purposely leave C and λ unspecified. The precise nature of λ does not play any role except for defining Ω. A dynamical theory will provide the criteria to decide whether or not a quadruple is in B_1. We do assume that if for some $\lambda > 0$, (T, C, λ, S) is in B_1 then so is (T, C, μ, S) for each $\mu \in [0, \infty)$ (see Section 3).

Furthermore, we say that the single beam b propagating in the tube T was created on the space-like surface S. This could describe, for instance, the solution of a hyperbolic partial differential equation corresponding to Cauchy data of compact support on S and propagating into T, according to Huyghens's principle. For a given element $b \in B_1$, we will write T_b, C_b, λ_b, S_b for the elements of $\mathscr{T}$, $\mathscr{C}$, $[0, \infty)$, $\mathscr{S}$, respectively, for which $b = (T_b, C_b, \lambda_b, S_b)$. We assume that $(T, C, 0, S) \in B_1$ for each $T \in \mathscr{T}$, $C \in \mathscr{C}$, and $S \in \mathscr{S}$. Thus $\lambda_\Omega = 0$ while T_Ω, C_Ω and S_Ω are arbitrary elements of their respective sets.

We now come to the relationship between interaction regions and elements of B_1. We say that an element $b \in B_1$ *enters* $R = (T, S_1, S_2) \in \mathscr{R}$ iff

(1) $T_b \cap S_1 \subseteq \mathrm{Beg}(R)$.
(2) $T_b \cap S_b < T_b \cap S_1$.

Condition (1) means that the trajectory describing b enters the region R while condition (2) means that b was created before reaching R.

We say that an element $b \in B_1$ *leaves* $R = (T, S_1, S_2) \in \mathscr{R}$ iff

$$S_b \cap T_b \subseteq \mathrm{End}(R)$$

This means that the state b was created on the back surface of R. It follows quite easily that if b leaves R then $S_b \cap T_b = S_2 \cap T_b$.

A collection $\{b_1, \ldots, b_n\}$ of elements of B_1 is said to *meet in R*, $R \in \mathscr{R}$, iff

(1) $b_i \neq b_j$ $(i,j = 1, \ldots, n)$.
(2) $b_i \neq \Omega$ $(i = 1, \ldots, n)$.
(3) Either all b_i, $i = 1, \ldots, n$, enter R or all b_i, $i = 1, \ldots, n$, leave R.

We are now in a position to define a beam which is supposed to represent a truly realizable state of the physical universe. A *beam* is a collection $(b_1, \ldots, b_n)$ of elements of B_1 such that there is an $R \in \mathscr{R}$ satisfying

(1) $(b_1, \ldots, b_n)$ meet in R.
(2) For all $R' \in \mathscr{R}$ disjoint from R, b_i and b_j do not meet in R' for any $i, j = 1, \ldots, n$.

The second condition means that the single beams comprising a beam interact once, and once only. We denote the collection of all beams plus the vacuum Ω as B.

This is not quite enough to describe actual experiments, for several beams (in the above sense) may coexist during a certain period of time. We therefore introduce

$$B^k = B \times \cdots \times B, \ k \text{ factors for } k \geqslant 1$$

and

$$B^{(n)} = \bigcup_{k=1}^{n} B^k$$

We note that

$$B = B^1 = B^{(1)}$$

As an example, $b \in B^{(2)}$ is either a beam or a pair of beams.

We distinguish certain subsets of B as follows. The *ingoing beams* for a region $R \in \mathscr{R}$ are

$$B_R^{\mathrm{in}} = \{b \in B | \quad \text{If } b = (b_1, \ldots, b_n), \text{ then } b_1, \ldots, b_n \text{ all enter } R\}$$

Similarly, the *outgoing beams* for R are

$$B_R^{\mathrm{out}} = \{b \in B | \quad \text{If } b = (b_1, \ldots, b_n), \text{ then } b_1, \ldots, b_n \text{ all leave } R\}$$

We note that

$$\Omega \in B_R^{\substack{\mathrm{in}\\\mathrm{out}}}$$

for each interaction region R.

Further, B_R^{in} (and B_R^{out}) have a natural cone structure.

We define the operations as follows:

(1) For $b=(T,C,\lambda,S)\in B_1\cap B_R^{\text{in}}$, we let
$$\mu b=(T,C,\mu\lambda,S)\quad\text{for}\quad \mu\geqslant 0$$
(2) For $b=(b_1,\ldots,b_n)\in B_R^{\text{in}}$, we let
$$\mu b=(\mu b_1,\ldots,\mu b_n)\quad\text{for}\quad \mu\geqslant 0$$
(3) For $b\in B_R^{\text{in}}$, we let
$$b+\Omega=b$$
(4) For $b\in B_R^{\text{in}}\cap B_1-\Omega$ and $c=(c_1,\ldots,c_n)\in B_R^{\text{in}}-\Omega$, we let
$$b+c\begin{cases}=(b,c_1,\ldots,c_n) & \text{if } b\neq c_i,\quad i=1,\ldots,n\\ =(c_1,\ldots,2c_i,\ldots,c_n) & \text{if } b=c_i\end{cases}$$
(5) For $b=(b_1,\ldots,b_n)\in B_R^{\text{in}}-\Omega$ and $c\in B_R^{\text{in}}$, we let
$$b+c=b_1+(b_2+(\cdots(b_n+c)\cdots)$$

It is obvious that B_R^{in} is a cone with neutral element Ω. We note that we may form the *real* vector spaces
$$V_R^{\substack{\text{in}\\ \text{out}}}=B_R^{\substack{\text{in}\\ \text{out}}}-B_R^{\substack{\text{in}\\ \text{out}}}$$
but we feel that the new elements have no physical interpretation.

Given an element $b\in B_1$, we can choose space-like surfaces S' and S'' such that
$$S'\cap T_b<S_b\cap T_b<S''\cap T_b$$
Then obviously b enters $R''=(T_b,S_b,S'')$ and leaves $R'=(T_b,S',S_b)$. Thus
$$b\in B_{R''}^{\text{in}}\cap B_{R'}^{\text{out}}$$
and so
$$B_1\subseteq\bigcup_{R\in\mathscr{R}}B_R^{\text{in}}\qquad\text{and}\qquad B_1\subseteq\bigcup_{R\in\mathscr{R}}B_R^{\text{out}}$$

Also, as long as a beam enters a region $R\in\mathscr{R}$, its properties in the region R do not depend on the epoch when it was created. We introduce, therefore, an equivalence relation $\approx$ on B_1 defined by:

For any pair $b,c\in B_1$, we say $b\approx c$ iff
$$T_b=T_c,\qquad C_b=C_c\qquad\text{and}\qquad \lambda_b=\lambda_c$$
This relation is then extended to B in an obvious manner.

2.3. *Interactions*

So far we have described interaction regions and beams which may enter or leave them. It remains to describe the concept of an experiment. The most general experiment consists of several pieces of equipment, located in a finite region of space-time, which transform an in-state into an out-state. Within the confines of the equipment, the original in-beam may be split into several disjoint beams (as in a scattering experiment, for example). If so, several counters may be placed within the apparatus in order to analyze these beams separately. Thus, the output of the experiment may also contain a finite number of numerical data (non-negative numbers). For a non-negative integer n, we therefore define an *n-channel experiment located in the region R*, denoted by $e_R{}^n$, to be a map $B \to B^{(2)} \times (\mathbb{R}^+)^n$ with the following properties [we will use the following notation: if $e_R{}^n(b) = (b', \beta)$, then $b' = \pi_1 e_R{}^n(b)$ and $\beta = \pi_2 e_R{}^n(b)$].

(1) If $b \in B_R^{\text{in}}$, then

$$e_R{}^n(b) = (b', \beta)\ b' \in B_R^{\text{out}},\ \beta \in (\mathbb{R}^+)^n$$

(2) In particular, $e_R{}^n(\Omega) = (b', (0, \ldots, 0))$.

(3) Suppose $b = (b_1, \ldots, b_N) \notin B_R^{\text{in}}$.

(*a*) If $b_i \notin B_R^{\text{in}}$, $i = 1, \ldots, N$, then

$$e_R{}^n(b) = (b, (0, \ldots, 0))$$

(*b*) If $b_{i_1}, \ldots, b_{i_k} \in B_R^{\text{in}}$ and $b_{i_{k+1}}, \ldots, b_{i_N} \notin B_R^{\text{in}}$, then

$$e_R{}^n(b) = ((\pi_1 e_R{}^n(b_{i_1}, \ldots, b_{i_k}), (b_{i_{k+1}}, \ldots, b_{i_N}), \pi_2 e_R{}^n(b_{i_1}, \ldots, b_{i_k}))$$

which is an element of $B^2 \times (\mathbb{R}^+)^n$.

(4) For $b \in B_R^{\text{in}}$, $e_R{}^n(b)$ depends only on the equivalence class of $b \bmod \approx$.

We denote by $A_R{}^n$ the collection of all n-channel experiments located in R. We define

$$A_R = \bigcup_{n=0}^{\infty} A_R{}^n$$

as the collection of all experiments located in R and finally

$$A = \bigcup_{R \in \mathscr{R}} A_R$$

as the collection of all experiments.

Several remarks are now in order. First, the symbols $e, f, \ldots$ always refer to a *specific* piece of equipment located in a region R. The word 'equipment' should be understood in a very wide sense. For instance, it may consist of only a magnetic field in R. It may even be *empty*, i.e. no equipment at all. Such a situation might describe the spontaneous decay of an unstable particle or the scattering between two particles.

We should like to remark that the numbers produced by the experiment $e_R{}^n$ are just that: numbers. A counter registered a number n or a spark chamber took pictures with n tracks. It is up to a dynamical theory, for example, to convert these numbers into dynamical variables such as energy and momentum or to infer from these numbers the existence or non-existence of particles with certain masses and spins. In fact, the entire superstructure of elementary and/or composite particles is just one interpretation of the numbers produced by the experiments.

When $b \in B_R^{\text{in}}$ we see that $e_R{}^n$ yields an element $b' \in B_R^{\text{out}}$. This element may be the vacuum, since $\Omega \in B_R^{\text{out}}$. Thus $e_R{}^n$ may 'destroy' the incoming beam and emit nothing; we leave this possibility open. Often, though, b' will be a multiple beam, i.e. $b' \notin B_1$. Let $b' = (b_1', \ldots, b_n')$ be the out beam. We may think of the single beams b_i', $i = 1, \ldots, n$, as only being 'part of the whole': correlations among the various single beams surely exist (imposed by various conservation laws, for example). One may further analyze one or several of these single-beams in the remote future (remote in the sense that measurements made on a beam immediately after its creation are part of the same experiment that created it). This analysis may infer some properties of the other single beams making up the complete out-state b'. An example of this situation is the celebrated Einstein–Rosen–Podolsky paradox (Einstein *et al.*, 1935; Bohm & Aharonov, 1957).

When $b \notin B_R^{\text{in}}$, one must distinguish two cases. If none of the constituents b_i of b enters R, the experiment $e_R{}^n$ cannot detect such a beam; it therefore leaves the state untouched and registers nothing. On the other hand, it may happen that a subset of the constituent single beams enters R while the rest passes nearby. In this case, the output of $e_R{}^n$ indeed consists of *two* beams: the one leaving the experiment and the remainder of the original beam. The numbers produced by the experiment $e_R{}^n$ are the same set of numbers as would be produced by $e_R{}^n$ applied only to the part of the beam entering R. By looking at a single state, which is a mixture, it is impossible to tell whether it is a true mixture (incoherent superposition) or an improper mixture (the restriction of a pure state of a bigger object) in the terminology of d'Espagnat (1965). In our language, one cannot tell whether a beam is an isolated single beam or 'part' or a multiple beam. Thus the numbers must be the same.

Also, the two beams in the output may or may not be correlated. The correlations persist in the spontaneous decay of an unstable particle produced in the first experiment. They may be destroyed (the so-called 'destruction of relative phases') as exemplified by a two-slit interference experiment (Feynman *et al.*, 1963) or a Stern–Gerlach experiment (Gottfried, 1966) where one analyzes one of the beams. One need not distinguish these cases as only an experiment which measures *all of the single beams* can detect the correlations (cf. d'Espagnat, 1965).

We do not require the map $b \to b'$ to be linear. Linearity does not play any role in the present formalism, since it is totally uncommitted to any dynamical theory. Since Wigner (1963) has argued that measurement theory might be incompatible with linearity, we allow for this possibility.

Of course, not all maps $B \to B^{(2)} \times (\mathbb{R}^+)^n$ will be experiments. It is part of a particular dynamical theory to tell which maps are in fact experiments and to predict the numerical values $\pi_2 e_R{}^n(b)$ for a given experiment and beam.

For $n = 0$, the above definition becomes simpler. A 0-channel experiment is one that yields an out-state but no data. Thus, it is a *preparation*. We note that $A_R{}^n$ may be canonically embedded in $A_R{}^0$ by merely omitting the data. Precisely, if $e_R{}^n \in A_R{}^n$ we associate to it $e_R{}^0 \in A_R{}^0$ by $e_R{}^0(b) = \pi_1 e_R{}^n(b)$ for all $b \in B$.

On the set of experiments A we may define a natural composition. One can perform two experiments in succession, letting the out-state of the first be the in-state of the second. The usual notion of repeated experiments is not allowed, as no beam which leaves an experiment can ever enter it again. We need to distinguish the cases when the first experiment 'sees' the entire beam and when it 'sees' only part of it (i.e., its output is in B or B^2). In the latter case, either (but not both) of the beams may enter the second experiment. By performing one experiment after another, we obviously receive two collections of numerical data. Modeling our choice on coincidence experiments, we feel that the output of the total operation is the output of the second experiment, as long as the first experiment registered something (i.e. its numerical result was anything but 0). In this latter case, the entire operation should report the number 0. Formally, then, let $e_R{}^n \in A_R{}^n$ and $f_S{}^m \in A_S{}^m$. Then the *composition of the two experiments*, denoted by $f_S{}^m \circ e_R{}^n$, is a map $B \to B^{(3)} \times (\mathbb{R}^+)^m$ defined by the following relations [we will use the following notation: if $f_S{}^m \circ e_R{}^n(b) = (b', \beta)$, then $b' = \rho_1(f_S{}^m \circ e_R{}^n(b))$ and $\beta = \rho_2(f_S{}^m \circ e_R{}^n(b))$].

(1) If $\pi_1 e_R{}^n(b) \in B$, then

$$\rho_1(f_S{}^m \circ e_R{}^n(b)) = \pi_1 f_S{}^m(\pi_1 e_R{}^n(b))$$

$$\rho_2(f_S{}^m \circ e_R{}^n(b)) = \begin{cases} (0,\ldots,0) & \text{if } \pi_2 e_R{}^n(b) = (0,\ldots,0) \\ \pi_2 f_S{}^m(\pi_1 e_R{}^n(b)) & \text{otherwise} \end{cases}$$

(2) If $\pi_1 e_R{}^n(b) = (c_1, c_2) \in B^2 - B$, then c_1 and c_2 cannot both be in B_S^{in}. Let $c_2 \notin B_S^{\text{in}}$. Then

$$\rho_1(f_S{}^m \circ e_R{}^n(b)) = (\pi_1 f_S{}^m(c_1), c_2) \in B^{(3)}$$

$$\rho_2(f_S{}^m \circ e_R{}^n(b)) = \begin{cases} (0,\ldots,0) & \text{if } \pi_2 e_R{}^n(b) = (0,\ldots,0) \\ \pi_2 f_S{}^m(c_1) & \text{otherwise} \end{cases}$$

Obviously, the composition of two experiments is not an experiment. Also, we note that $\circ$ is not commutative: $f_S \circ e_R \neq e_R \circ f_S$ in general. Lastly, we see that $\rho_2(e_R \circ e_R) \equiv (0,\ldots,0)$.

We may go further and define composed experiments by induction:

$$A^{(2)} = A \circ A$$

$$A^{(n)} = A \circ A^{(n-1)}$$

Note that $A^{(n)}$ maps B into $B^{(n+1)}$. Letting

$$\tilde{A} = \bigcup_{n=1}^{\infty} A^{(n)}$$

be the collection of all multiple experiments, we see that composition is a map $\tilde{A} \times \tilde{A} \to \tilde{A}$. Since $A^{(n)} \circ A^{(m)} \subset A^{(n+m)}$, $\tilde{A}$ is a *graded algebra.*

3. *Axioms*

The first two axioms deal with the single beams B_1 and have already been mentioned. It is the task of a dynamical theory to decide which sets $C \in \mathscr{C}$ describe actual physical 'particles'. Further, for a realizable $C \in \mathscr{C}$ only certain tubes T and original surfaces S may be allowed by a given theory. We, however, assume that the following hold (which might possibly limit the choice of dynamics).

Axiom 1

For each $T \in \mathscr{T}$, $C \in \mathscr{C}$, and $S \in \mathscr{S}$, we have $(T, C, 0, S) \in B_1$.

Axiom 2

For each $T \in \mathscr{T}$, $C \in \mathscr{C}$, and $S \in \mathscr{S}$, if there is a $\lambda > 0$ such that $(T, C, \lambda, S) \in B_1$, then *$(T, C, \mu, S) \in B_1$ for all $\mu \in [0, \infty)$.*

This axiom means that if a certain state exists, then it can be made more or less intense.

The second group of axioms deals with the relationship between experiments and beams.

Axiom 3

For each $R \in \mathscr{R}$ and $b \in B_R^{\text{in}}$, there is some $e_R \in A_R$ such that $\pi_1 e_R(b)$ is not equivalent to b mod $\approx$ or that $\pi_2 e_R(b) \neq (0, \ldots, 0)$.

This axiom means that every beam is affected and/or detectable by some experiment in each region that it enters.

Axiom 4

For each $R \in \mathscr{R}$ and $b \in B_R^{\text{out}}$, there is some $e_R \in A_R$ and $c \in B_R^{\text{in}}$ such that $\pi_1 e_R(c) = b$.

This axiom means that every out-state is created by some experiment.

By what has been assumed so far, we may have an empty theory. We therefore have the following existence axiom.

Axiom 5

For each $T \in \mathscr{T}$ and $S \in \mathscr{S}$, there is a $C \in \mathscr{C}$ and $\lambda > 0$ such that $(T, C, \lambda, S) \in B_1$.

This means that given any tube, there is a physical state (other than the vacuum) propagating in its which was created on any specified space-like surface. We note that Axiom 3, together with Axiom 5, implies the existence of many non-trivial experiments. In physical terms, this axiom means that there exist universal beams capable of traveling in any prescribed way (at a speed not in excess of the speed of light). Such a requirement is inspired, among others, by the use of electron beams and photon beams as quasi-universal probes, both at the atomic and nuclear level.†

4. *Causality Structure*

Instead of the ordering of points used in the usual discussion of causality, we here shall deal with an order relation between interaction regions. Given two *disjoint* elements of $\mathscr{R}$, say $R=(T^R, S_1{}^R, S_2{}^R)$ and $S=(T^S, S_1{}^S, S_2{}^S)$, we say that (see Fig. 1)‡

(1) R is *earlier* than S, written $R < S$, iff $\mathrm{Beg}(S) \cap V^+(\mathrm{End}(R)) \neq \varnothing$.
(2) R is *later* than S, written $R > S$, iff $\mathrm{Beg}(R) \cap V^+(\mathrm{End}(S)) \neq \varnothing$.
(3) R is *space-like* to S, written $R \rhd\!\lhd S$, iff both $\mathrm{Beg}(S) \cap V^+(\mathrm{End}(R)) = \varnothing$ and $\mathrm{Beg}(R) \cap V^+(\mathrm{End}(S)) = \varnothing$.

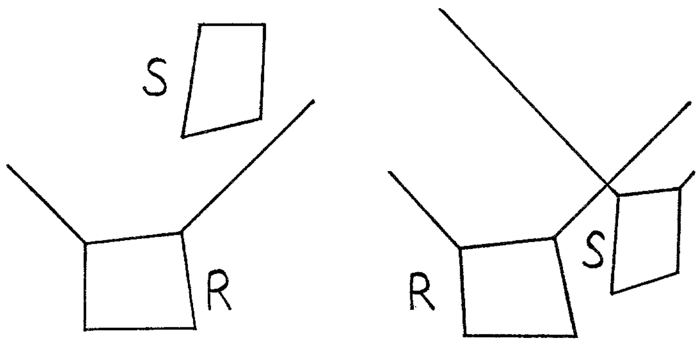

Figure 1—First diagram: $R < S$. Second diagram: $R \rhd\!\lhd S$.

By construction, *the three possibilities are mutually exclusive and exhaustive*. The relation '<' is, however, *not* a partial order, for it need not be transitive. Due to the finite thickness of interaction regions, it is possible that $R < S$, $S < U$ and $R \rhd\!\lhd U$ for regions $R, S, U \in \mathscr{R}$ (see Fig. 2). The

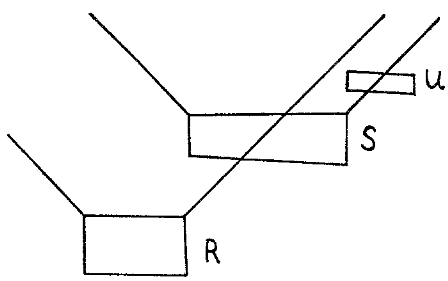

Figure 2

† See, for instance, Mott & Massey (1965) and Hofstadter (1963).
‡ $V^+(A)$ = open forward light cone generated by $A\{x \mid (\exists y \in A)\ (x-y)^2 > 0,\ x^\circ - y^\circ > 0\}$ (we use the metric $(+,-,-,-)$).

set $\mathscr{R}$ is, however, directed in both directions by '<'. That is, given any pair R_1 and R_2 in $\mathscr{R}$, there exists $R_3 \in \mathscr{R}$ earlier than both R_1 and R_2, and there exists $R_4 \in \mathscr{R}$ later than both R_1 and R_2.

We now relate the order relation of interaction regions to the experiments. To do this, we introduce the concept of transparency. We say that the experiment $f_S \in A_S$ is *transparent* to $e_R \in A_R$ iff for each $b \in B$ which satisfies $\pi_1 e_R(b) \neq \Omega$, we have

$$f_S \circ e_R(b) = (\pi_1 e_R(b), (0, \ldots, 0))$$

Less formally, suppose a beam b passed through the experiment e_R to produce the out-state $b' \neq \Omega$. Then f_S is transparent to e_R if the out-beam from f_S is still b' and no coincidences between e_R and f_S were recorded. We note that for every $e \in A$, e is transparent to e. The class A_S is transparent to the class A_R iff f_S is transparent to e_R for each $e_R \in A_R, f_S \in A_S$. (Recall that A_R is the set of all experiments in the region R.)

Proposition 4.1.

Let $R, S \in \mathscr{R}$. Then $R < S$ iff A_S is not transparent to A_R. In the latter case, we write $A_R < A_S$.

Proof: Suppose $R < S$. Then there exists a world line W from the interior of End(R) to the interior of Beg(S). Taking a small convex neighborhood of W, we see that there is a tube T such that $T \cap \text{End}(R) < T \cap \text{Beg}(S)$, where '<' refers to the Minkowski time ordering. By Axiom 5, there is a $b \in B$ such that $b = (T, C, \lambda, S_2{}^R)$. Then $b \in B_R^{\text{out}} \cap B_S^{\text{in}}$ by construction. By Axiom 4, there is an experiment $e_R \in A_R$ and a $c \in B_R^{\text{in}}$ such that $\pi_1 e_R(c) = b$. Further, by Axiom 3, there is an experiment $f_S \in A_S$ such that $\pi_1 f_S(b)$ is not equivalent to b mod $\approx$ or $\pi_2 f_S(b) \neq (0, \ldots, 0)$. In either case, f_S is not transparent to e_R.

Conversely, if A_S is not transparent to A_R, then there is a beam $c \in B$ and experiments $f_S \in A_S$ and $e_R \in A_R$ such that $\pi_1 e_R(c) = b \neq \Omega$ and $f_S \circ e_R(c) \neq (b, (0, \ldots, 0))$. Hence, $b \in B_R^{\text{out}} \cap B_S^{\text{in}}$, and so T_b is a tube connecting End(R) to Beg(S). Since T_b is composed of geodesics, it is trivial that $\text{Beg}(S) \cap V^+(\text{End}(R)) \neq \varnothing$. Q.E.D.

We note that the proof above also showed that $A_R < A_S$ iff there exists a $b \in B$, $b \neq \Omega$, and $b \in B_R^{\text{out}} \cap B_S^{\text{in}}$.

Corollary 4.2

Suppose $R, S \in \mathscr{R}$, Then $R < S$ implies that A_R is transparent to A_S.

Proof: Since $R < S$, it is false that $S < R$. Now use Prop. 4.1.

Corollary 4.3

Suppose $R, S \in \mathscr{R}$. Then $R \rhd\lhd S$ iff A_R is transparent to A_S and A_S is transparent to A_R. In the latter case, we write $A_R \rhd\lhd A_S$.

Thus, we have a correspondence between the causality structure of $\mathscr{R}$ and the transparency structure of A. The next step is to relate these structures to the causality structure of the underlying Minkowski space. To each point $p \in \mathscr{M}$ we associate a family of regions $R_p^{(n)} \in \mathscr{R}$ whose intersection is $\{p\}$, i.e.

$$\bigcap_n R_p^{(n)} = \{p\}$$

One may obviously consider only families whose elements are ordered by inclusion, i.e.

$$R_p^{(n)} \subset R_p^{(m)} \quad \text{if } n > m$$

By a family we shall mean a totally ordered family. For points $p, q \in \mathscr{M}$ we shall use the following notation:

(1) $p <_T q$ for $(q-p)^2 > 0$ and $q^0 - p^0 > 0$.
(2) $p <_L q$ for $(q-p)^2 = 0$ and $q^0 - p^0 > 0$.
(3) $p \rhd\!\lhd\, q$ for $(q-p)^2 < 0$.

We then have the following result.

Proposition 4.4

Suppose p and q are distinct points in $\mathscr{M}$. Then:

(1) $p <_T q$ iff for all families $R_p^{(n)}$, $S_q^{(n)}$ converging to p and q, respectively, $R_p^{(n)} < S_q^{(n)}$ eventually.
(2) $p \rhd\!\lhd\, q$ iff for all families $R_p^{(n)}$, $S_q^{(n)}$ converging to p and q, respectively, $R_p^{(n)} \rhd\!\lhd\, S_q^{(n)}$ eventually.
(3) $p <_L q$ iff there exist families $R_p^{(n)}$, $S_q^{(n)}$, $\tilde{R}_p^{(n)}$, and $\tilde{S}_q^{(n)}$ converging to p, q, p, and q, respectively, such that $R_p^{(n)} < S_q^{(n)}$ eventually and $\tilde{R}_p^{(n)} \rhd\!\lhd\, \tilde{S}_q^{(n)}$ eventually.

Proof: Let $p <_T q$ and $R_p^{(n)}$ and $S_q^{(n)}$ be families converging to p and q, respectively. Since $p <_T q$, $q \in V^+(p)$ and so $S_q^{(n)} \subseteq V^+(p)$ eventually (recall that $V^+(p)$ is open). Let $V^-(n) \equiv V^-(\mathrm{Beg}(S_q^{(n)})) =$ open backward cone generated by $\mathrm{Beg}(S_q^{(n)})$. Then $p \in V^-(n)$ and so $R_q^{(n)} \subseteq V^-(n)$ eventually. Thus $V^+(\mathrm{End}(R_p^{(n)})) \cap V^-(n) \neq \varnothing$ eventually which trivially implies $R_p^{(n)} < S_q^{(n)}$ (see Fig. 3a).

Let $p \rhd\!\lhd\, q$ and $R_p^{(n)}$ and $S_q^{(n)}$ be families converging to p and q, respectively. Then an argument similar to that above shows that $V^+(\mathrm{End}(R_p^{(n)})) \cap V^-(n) = \varnothing$ eventually which means that $R_p^{(n)} \rhd\!\lhd\, S_q^{(n)}$, eventually (see Fig. 3b).

Finally, let $p <_L q$. Let

$$R_p^{(n)} = \left\{x \middle| \ |x^0 - p^0| \leqslant \frac{1}{n} \quad \text{and} \quad |\vec{x} - \vec{p}| \leqslant \frac{2}{n}\right\}$$

and $S_q^{(n)}$ be the same with q replacing p. Let

$$\tilde{R}_p^{(n)} = \left\{x \middle| \ |x^0 - p^0| \leqslant \frac{2}{n} \quad \text{and} \quad |\vec{x} - \vec{p}| \leqslant \frac{1}{n}\right\}$$

and $\tilde{S}_q^{(n)}$ be the same with q replacing p. Then we have

$$R_p^{(n)} < S_q^{(n)} \qquad \text{and} \qquad \tilde{R}_p^{(n)} \rhd\!\lhd \tilde{S}_q^{(n)}$$

eventually (see Fig. 3c and Fig. 3d).

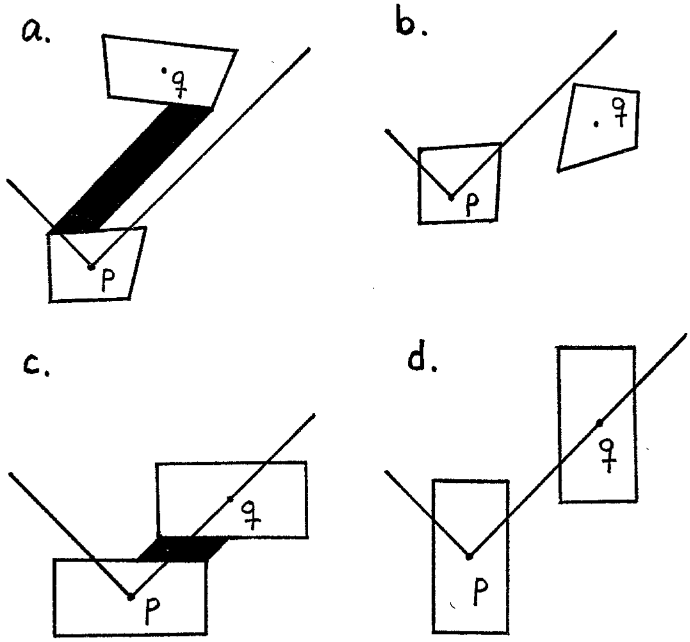

Figure 3—The shaded area is the light-like part of $V^+(\text{End}(R_p^{(n)})) \cap V^-(n)$.

We have now established that a collection of five statements (include $q <_T p$ and $q <_L p$) which are mutually exclusive and exhaustive imply five statements which are mutually exclusive and exhaustive. Thus, we are done. Q.E.D.

Combining the last few results, we get

Proposition 4.5

Let p and q be distinct points in $\mathcal{M}$. Then:

(1) $p <_T q$ iff for all families $R_p^{(n)}$, $S_q^{(n)}$ converging to p and q, respectively, $A_{R_p^{(n)}} < A_{S_q^{(n)}}$ eventually.

(2) $p \rhd\!\lhd q$ iff for all families $R_p^{(n)}$, $S_q^{(n)}$ converging to p and q, respectively, $A_{R_p^{(n)}} \rhd\!\lhd A_{S_q^{(n)}}$ eventually.

(3) $p <_L q$ iff there exist families $R_p^{(n)}$, $S_q^{(n)}$, $\tilde{R}_p^{(n)}$, and $\tilde{S}_q^{(n)}$ converging to p, q, p and q, respectively, such that $A_{R_p^{(n)}} < A_{S_q^{(n)}}$ and $A_{\tilde{R}_p^{(n)}} \rhd\!\lhd A_{\tilde{S}_q^{(n)}}$ eventually.

5. *Poincaré Invariance*

Poincaré transformations are defined as elements of the isometry group of $\mathscr{M}$. In the usual formulations of measurement theory, they are realized in the Hilbert space of states as unitary or anti-unitary transformations. As we have no Hilbert space of states, we must define Poincaré transformations on our objects geometrically. We shall in the following consider only the time-preserving subgroup of the Poincaré group, denoted by $P^{\uparrow}$. (One can easily consider time-reversal with but minor changes in the following.)

We let $\Lambda \in P^{\uparrow}$ be arbitrary. Then, by definition, $p <_{T,L} q$ iff $\Lambda p <_{T,L} \Lambda q$ and $p \rhd\lhd q$ iff $\Lambda p \rhd\lhd \Lambda q$. Hence, if $T \in \mathscr{T}$ is a tube, then so is ΛT and $\Lambda^{-1} T$. Similar statements hold for space-like surfaces. Hence $R \in \mathscr{R}$ iff $\Lambda R \in \mathscr{R}$. Further, Λ preserves the relations '$<$' and '$\rhd\lhd$' on $\mathscr{R}$. If $b \in B_1$, then Λb is the beam in the transformed tube created on the transformed surface. *A priori* this need not be an element of B_1. So we assume:

Axiom 6

If $b = (T, C, \lambda, S) \in B_1$, *then* $\Lambda b = (\Lambda T, C, \lambda, \Lambda S) \in B_1$ for any $\Lambda \in P^{\uparrow}$.

We extend this action to the beams by letting Λ act on the components, i.e. if $b = (b_1, \ldots, b_n) \in B$, then $\Lambda b = (\Lambda b_1, \ldots, \Lambda b_n)$ and, from Axiom 6, $\Lambda b \in B$. The following results are rather trivial consequences.

Proposition 5.1

(1) $\Lambda \Omega = \Omega$ for all $\Lambda \in P^{\uparrow}$.
(2) $b_1 \approx b_2$ iff $\Lambda b_1 \approx \Lambda b_2$ for $b_1, b_2 \in B$, $\Lambda \in P^{\uparrow}$.
(3) If $b \in B$ satisfies $\Lambda b = b$ for all $\Lambda \in P^{\uparrow}$, then $b = \Omega$.
(4) Suppose $b \in B_1$. Then b leaves (enters) R iff Λb leaves (enters) ΛR for $R \in \mathscr{R}$, $\Lambda \in P^{\uparrow}$.
(5) $\Lambda(B_R^{\text{in}}) = B_{\Lambda R}^{\text{in}}$ for $\Lambda \in P^{\uparrow}$.
(6) $\Lambda(B_R^{\text{out}}) = B_{\Lambda R}^{\text{out}}$ for $\Lambda \in P^{\uparrow}$.

As to the experiments, the situation is more subtle. There are two types of invariance which must be carefully distinguished. One is the invariance of space-time under $P^{\uparrow}$ (so-called homogeneity of space-time); this is already embodied by taking $\mathscr{M}$ as the underlying space-time manifold. The other type of invariance under $P^{\uparrow}$ is that of a particular dynamical theory. Of course, experiments are designed to test those invariances. The important point is that they always test *both*, for no experiment can distinguish between them. One must therefore *postulate* the invariance of space-time. Then, and only then, can experiments be used to test the invariance of a particular theory. A consequence of this fact is the equivalence of the active and the passive interpretation of symmetries (Antoine, 1969). Keeping this in mind, we may now define the transformation properties of experiments.

Suppose $e_R \in A_R$ is a given piece of equipment located in R. Then Λe_R is *uniquely* defined to be the transformed (i.e. translated, rotated, boosted, etc.) apparatus. This is not the same thing as the original apparatus put in the region ΛR, for this may even be impossible. The postulate of the invariance of space-time under $P\uparrow$ would then imply that Λe_R is an experiment in ΛR and that its out-beam is the transformed out-beam. Thus, we now assume:

Axiom 7

Suppose $e_R \in A_R$, $b \in B$, *and* $e_R(b) = (b', \beta)$. *Then, for any* $\Lambda \in P^\uparrow$, $\Lambda e_R \equiv (\Lambda e)_{\Lambda R} \in A_{\Lambda R}$ *and further*, $\Lambda e_R(\Lambda b) = (\Lambda b', \beta')$. *Also*, $\Lambda A_R = A_{\Lambda R}$.

We note that the numbers β and β' may be different. The Poincaré invariance of the theory precisely means that these numbers are the same. We do not require that the dynamical theory be Poincaré invariant. We observe from the above that the elements of $A_R{}^0$ are incapable of testing the invariance of a theory; one needs genuine measurements to do so.

We note that the transparency relation behaves very simply under $P^\uparrow$. We saw (in the remark following Proposition 4.1) that $A_R < A_S$ was equivalent to the existence of a $b \in B$, $b \neq \Omega$ such that $b \in B_R^{\text{out}} \cap B_S^{\text{in}}$. Hence, Proposition 5.1, parts (5) and (6), and Axiom 7 above immediately yield:

Proposition 5.2

Let $R, S \in \mathscr{R}$ and $\Lambda \in P^\uparrow$. Then $A_R < A_S$, $\Lambda A_R < \Lambda A_S$, and $A_{\Lambda R} < A_{\Lambda S}$ are equivalent.

This is quite reassuring in the light of Proposition 4.5. In fact, if a transformation Γ of $\mathscr{M}$ preserves the transparency relation then Γ almost is an element of $P^\uparrow$. Following Zeeman (1964), we define $G^\uparrow$ to be the group generated by $P^\uparrow$ and the dilations.

Theorem 5.3

Let $\Gamma: \mathscr{M} \to \mathscr{M}$ satisfying

(1) Γ is one to one.
(2) There is a $\delta > 0$ such that if $R \in \mathscr{R}$ and diameter† $(R) < \delta$, then $\Gamma(R)$ and $\Gamma^{-1}(R)$ are in $\mathscr{R}$.
(3) If $R, S \in \mathscr{R}$ each have diameter less than δ, then $A_R < A_S$ iff $A_{\Gamma(R)} < A_{\Gamma(S)}$.

Then $\Gamma \in G^\uparrow$.

Proof: Let p and q be distinct points of $\mathscr{M}$. Suppose $p <_T q$. Let $R^{(n)}_{\Gamma(p)}$ and $S^{(n)}_{\Gamma(q)}$ be families converging to $\Gamma(p)$ and $\Gamma(q)$, respectively. Then, eventually,

† Supremum of the Euclidean distances between points of R.

the diameters of these regions are smaller than δ. So, $\Gamma^{-1} R^{(n)}_{\Gamma(p)}$ and $\Gamma^{-1} S^{(n)}_{\Gamma(q)}$ are families converging to p and q, respectively. Thus

$$A_{\Gamma^{-1} R^{(n)}_{\Gamma(p)}} < A_{\Gamma^{-1} S^{(n)}_{\Gamma(q)}}$$

eventually and so

$$A_{R^{(n)}_{\Gamma(p)}} < A_{S^{(n)}_{\Gamma(q)}}$$

eventually. Hence $\Gamma(p) <_T \Gamma(q)$ by Proposition 4.1. Applying the argument to Γ^{-1} we get that $p <_T q$ iff $\Gamma(p) <_T \Gamma(q)$. Similarly, one may show that $p <_L q, p \rhd\!\lhd q, q <_T p$, and $q <_L p$ hold if and only if the corresponding statement for $\Gamma(p)$ and $\Gamma(q)$ holds. We may therefore appeal to Zeeman (1964) for the conclusion. Q.E.D.

This result is an operational version of Zeeman's theorem. As in the original theorem, neither continuity nor linearity are assumed *a priori*. Also, it is remarkable that not all relations between arbitrary experiments need be preserved in order that a transformation Γ belong to $G^{\uparrow}$; Γ need only preserve transparency relations, i.e. *null* results, between *local* experiments (δ may be arbitrarily small).

6. *Conclusion*

We should like to re-emphasize that the present formalism is completely independent of any dynamical theory. It is therefore compatible with quantized theories as well as classical ones, with linear theories as well as non-linear ones. It is, of course, true that orthodox quantum mechanics is linear. However, all interacting field equations are essentially non-linear. An energetic particle entering an experiment can produce a large number of particles and so one may question linearity as soon as pair production processes are present. In other words, it is possible that operational linear theories are valid approximations to reality only at low energy. Furthermore, for completely different reasons, linearity might be incompatible with measurement theory itself, as argued by Wignes (1963). Thus we feel justified in leaving the door open to non-linear theories as well.

Symbols	*Description*
$\mathscr{M}$	Minkowski space
$\mathscr{S}$	all space-like surfaces
$\mathscr{T}$	all (world) tubes
$\mathscr{R}$	all interaction regions
Beg(R)	beginning of interaction region R
End(R)	end of interaction region R
$\mathscr{C}$	an unspecified collection of quantum numbers
B_1	the single beams
Ω	vacuum

Symbols	*Description*
B	beams
B^k	Cartesian product of B with itself k-times
$B^{(n)}$	$\bigcup_{k=1}^{n} B^k$
B_R^{in}	all ingoing beams for region R
B_R^{out}	all outgoing beams for region R
$\approx$	an equivalence relation on the beams
$\mathbb{R}$	real numbers
$\mathbb{R}^+$	non-negative real numbers
π_1	a projection operator
π_2	a projection operator
$A_R{}^n$	all n-channel experiments in R
A_R	all experiments in R
A	all experiments
$f_S{}^m \circ e_R{}^n$	composition of two experiments
ρ_1	a projection operator
ρ_2	a projection operator
$A^{(n)}$	n-composed experiments
$\tilde{A}$	all multiple experiments
$\varnothing$	the empty set
$R < S$	region R is earlier than region S
$R > S$	region R is later than region S
$R \bowtie S$	region R is space-like to region S
$V^+(A)$	open forward light cone generated by A $= \{x \mid (\exists y \in A)(x-y)^2 > 0 \text{ and } x^0 - y^0 > 0\}$
$A_R < A_S$	A_S is not transparent to A_R
$A_R \bowtie A_S$	A_R is transparent to A_S and conversely
$p <_T q$	$(q-p)^2 > 0$ and $q^\circ - p^\circ > 0$
$p <_L q$	$(q-p)^2 = 0$ and $q^\circ - p^\circ > 0$
$p \bowtie q$	$(q-p)^2 < 0$
$P^\uparrow$	group of time preserving Poincaré transformations
$G^\uparrow$	group generated by $P^\uparrow$ and the dilations

Axioms

1. $(\forall T \in \mathscr{T})\,(\forall C \in \mathscr{C})\,(\forall S \in \mathscr{S})$
$$(T, C, 0, S) \in B_1$$
2. $(\forall T \in \mathscr{T})\,(\forall C \in \mathscr{C})\,(\forall S \in \mathscr{S})$
$$(\exists \lambda > 0)\,(T, C, \lambda, S) \in B_1 \Rightarrow (\forall \mu \geqslant 0)\,(T, C, \mu, S) \in B_1$$

3. $(\forall R \in \mathscr{R})\,(\forall b \in B_R^{\text{in}})\,(\exists e_R \in A_R)$
$$\pi_1 e_R(b) \not\approx b \text{ or } \pi_2 e_R(b) \not\approx (0,\ldots,0)$$
4. $(\forall R \in \mathscr{R})\,(\forall b \in B_R^{\text{out}})\,(\exists e_R \in A_R)\,(\exists c \in B_R^{\text{in}})$
$$\pi_1 e_R(c) = b$$
5. $(\forall T \in \mathscr{T})\,(\forall S \in \mathscr{S})\,(\exists C \in \mathscr{C})\,(\exists \lambda > 0)$
$$(T, C, \lambda, S) \in B_1$$
6. $(\forall \Lambda \in P^{\uparrow})\,(\forall b = (T, C, \lambda, S) \in B_1)$
$$\Lambda b \equiv (\Lambda T, C, \lambda, \Lambda S) \in B_1$$
7. (*a*) $(\forall e_R \in A_R)\,(\forall b \in B)\,(\forall \Lambda \in P^{\uparrow})$
$$e_R(b) = (b', \beta) \Rightarrow \Lambda e_R \equiv (\Lambda e)_{\Lambda R} \in A_{\Lambda R} \text{ and } \Lambda e_R(\Lambda b) = (\Lambda b', \beta')$$
(*b*) $(\forall R \in \mathscr{R})\,(\forall \Lambda \in P^{\uparrow})\,\Lambda A_R = A_{\Lambda R}$

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